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Analysis of Ranking Factors for a Risk Averse Investor in a Non-Gaussian World

In this paper the authors discuss the relations between measures of risk, utility functions, and ranking factors for portfolio selection. They prove an exact equivalence between Sharpe, Sortino, Omega, Kappa, and Stutzer rankings in the case of Gaussian distributions. They also derive an exact "corrected" Levy ranking formula that applies to a portfolio with non-Gaussian distributions when the investment amount is predetermined. All the results presented in this paper have been proven by explicit analytical calculations. For brevity and clarity, details of those calculations are omitted from the paper.

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INTRODUCTION

In this paper we discuss the relations between measures of risk, utility functions, and ranking factors for portfolio selection. Our focus is on portfolio selection and not on portfolio construction; therefore we identify a portfolio with its distribution of returns (Ortobelli *et.al.*, 2005). A portfolio may represent a single asset or other investment opportunity. We also distinguish two main cases: in Case #1 the investor makes a portfolio selection before deciding the quantity of money to allocate into the portfolio; in Case #2 the investor makes a portfolio selection after a fixed amount of money has been allocated for the investment. Although these two cases appear very similar they are not.

In Case #1 we conclude that, for portfolios with Gaussian distributed returns, the ranking schemes known as Sharpe, Sortino, Kappa, Omega, and Stutzer are all appropriate and equivalent because they produce the same relative rankings. For non-Gaussian distributed returns, the above rankings are not equivalent and correspond to different (and subjective) definitions of risk.

In Case #2 we conclude that, for portfolios with Gaussian distributed returns, only the Levy ranking is correct, and it corresponds to being rational and risk averse. The use of Sortino, Kappa, Omega, and Stutzer, in Case #2, would correspond to not being risk averse. For non-Gaussian distributed returns we have derived a "corrected" Levy ranking formula (Levy and Markowitz, 1979) that incorporates the effects of skewness and kurtosis of the distribution in the ranking measure:

$$[return] + m/2[risk] - m/6\,[risk]\;\;[skewness] + m/720\,[risk]\;\;[kurtosis] \quad (\text{“corrected” Levy}) \,, \tag{1}$$

(*[kurtosis]* here is the reduced kurtosis and it is zero for a Gaussian, m is the risk aversion parameter of the CARA utility function).

CASE #1

As pointed out by H. Markowitz (Markowitz 1952), in Case #1, the investor can make a selection without having to choose a utility function, although he must make a choice on how to measure risk. The choice of a utility function is necessary only to decide how much to allocate into the selected portfolio and how much to allocate into a risk-free asset (for example in the bank or in a U.S. Treasury bill). In Case #1 the investor would select the optimal portfolio by maximizing the ratio:

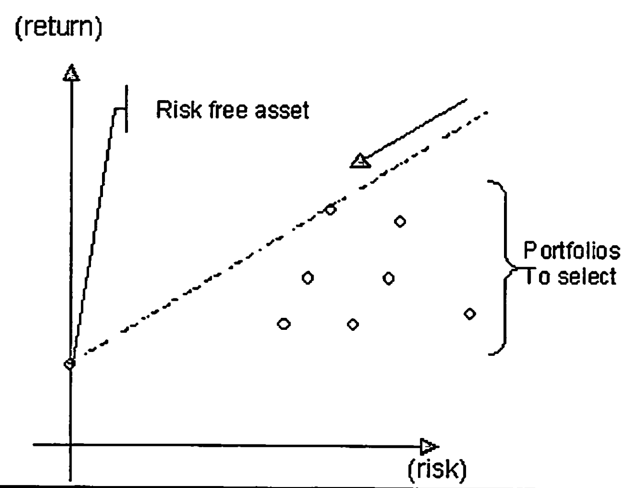
$$([return] - [benchmark])/[risk] \quad \text{(Sharpe ranking)} \quad . \tag{2}$$

Here *[return]* is the average return or the expected average return for the portfolio; *[benchmark]* is the risk-free rate. In a world were all portfolios are characterized by a Gaussian distribution of returns, the most common measure of *[risk]* is the standard deviation

$$[risk] = \text{mean of } (x[I] - [return]) \quad \text{(Risk)} \ , \tag{3}$$

where *x[I]* are historical returns. In such a world the above ranking formula takes the name of Sharpe ratio (Sharpe 1964). The Sharpe ratio is the correct ranking scheme in Case #1 for Gaussian portfolios. After the investor has selected the optimal portfolio, the investor

Figure 1: Portfolio Choice



is free to distribute the available funds between this portfolio and the risk-free asset. This requires a second choice. On a risk-return plane the set of available choices is represented by the Capital Allocation Line that passes through a point corresponding to the risk-free asset and the point corresponding to the optimal portfolio.

The figure shows a set of portfolios and a risk-free asset. In Case #1 the investor has to choose one portfolio and then combine it with the risk-free asset (dashed line).

This second choice is subjective and depends on the investor's own utility function, *i.e.,* how to translate return (or wealth) into satisfaction. According to utility theory (Neumann 1947, Nash 1950) an investor who makes choices compatible with a utility function is defined as "rational." An investor who makes choices compatible with a monotonic increasing utility function is defined "rational" and "risk averse." A common monotonic increasing utility function is the Constant Absolute Risk Aversion (CARA) utility function.

<u>Ranking Factors</u>

Various authors have proposed other measures of risk, for example downside risk

$$[downside\ risk] = \text{mean of } (x_i - [benchmark]) \ \ for\ x_i < [benchmark] \quad \text{(Downside Risk)}\ . \tag{4}$$

With this definition of risk the Sharpe ratio becomes the Sortino ratio (Sortino and von der Meer 1991).

We say that two ranking schemes are equivalent if and only if, for any set of portfolios, they generate the same relative ranking. In mathematical terms, two rankings, R_1 and R_2, are equivalent (DiPierro and Mosevitch 2004) if there is a monotonic increasing function h such that for every portfolio A the following relation holds:

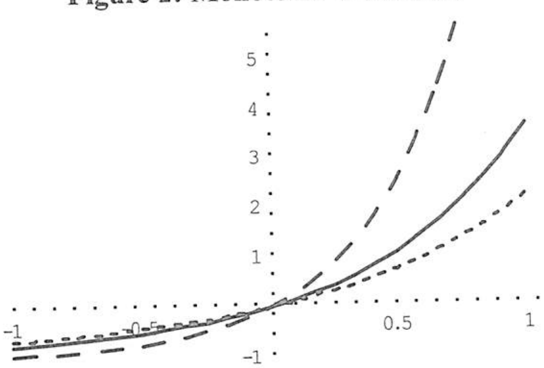
$$R_1(A)=h(R_2(A)) \quad \text{(Equivalence Relation)}\ . \tag{5}$$

The continuous line represents the value of the Sortino ratio as function of the Sharpe ratio for Gaussian portfolios; the large dashed line represents the value of the

Omega ratio as a function of the Sharpe ratio for Gaussian portfolio; the small dashed line represents the value of the Kappa ratio (n=3) as function of the Sharpe ratio for Gaussian portfolios. Notice that Kappa at n=1 is equivalent to Omega and Kappa at n=2 is equivalent to Sortino by definition. Notice that all these functions are monotonic and therefore correspond to equivalent rankings in the Gaussian case.

We have proven that in a Gaussian world, Sharpe (Sharpe 1964), Sortino (Sortino and Price 1994) (Sortino and Forsey 1996), Kappa (Kaplan and Knowles 2004), Omega (Sortino 2001, Kazemi *et.al.*, 2003), and Stutzer (Stutzer 2000), are all equivalent ranking schemes. In practical terms, in this case, this implies that if portfolio A and portfolio B are characterized by two Gaussian distributions and if Sharpe(A)>Sharpe(B) then Sortino(A)>Sortino(B), Kappa(A)>Kappa(B), Omega(A)>Omega(B), and Stutzer(A)>Stutzer(B). This is not surprising since Gaussian distributions depend on a single parameter: the width of the Gaussian distribution. The equivalence is shown in Figure 2. In the case of non-Gaussian distributions the ranking schemes are no longer equivalent because they follow from different definitions of risk. For a review see (Artzner and Delbaen, 2000).

Figure 2: Monotonic Functions



CASE #2

Case #2 is different since a predetermined amount of money has to be invested in the selected portfolio. The two choices of Case #1 are now combined in a single choice and it requires a utility function. As pointed out by Levy and Markowitz, in a world where all portfolios are characterized by a Gaussian distribution, a rational risk averse investor using CARA would rank the portfolios using

$$[return] - m/2\ [risk]^2 \quad \text{(Levy Ranking)} \quad . \tag{6}$$

Here m is a subjective risk aversion factor of order one. The larger m, the more *[risk]* is penalized. In this paper we refer to the latter ranking formula as Levy ranking.

In mathematical terms, given a utility function U, a portfolio distribution P, and a ranking R, we say that R corresponds to (or is compatible with) U if and only if there is a monotonic increasing function h such that

$$\int U(x)P(x)dx = h(R(P)) \quad \text{(Correspondence Relation)} \ . \tag{7}$$

In other words, a ranking R corresponds to a utility function U if and only if the weighted average of the utility U of all possible returns of portfolio P generates the same relative ranking as $R(P)$.

We have computed the above integral for various utility functions and we have been able to establish the following relations (See Table 1).

We discuss each of these relations one by one.

Theta Utility Function

An investor who uses the Sharpe ratio to rank portfolios

Table 1: Portfolio/Utility Function Relations

Portfolio	Ranking Scheme	Utility Function
Gaussian	Sharpe, Sortino, Kappa, Omega, Stutzer	θ-function
Gaussian	Levy	CARA
Non-Gaussian	"corrected" Levy	CARA
Non-Gaussian	Levy	CARA*
Non-Gaussian	Levy using [downside risk]	CARA**

in order to allocate a predetermined investment amount (our Case #2) is implicitly using the **θ** utility function to make his selection. The **θ**-function is always equal to +1 for positive returns and always equal to -1 for negative returns. This means that the investor values positive returns more than negative returns, but the investor does not value +100% more than +1%, nor -100% less than -1%. Therefore the investor who uses exclusively the Sharpe ratio to make his decision is rational but is not a risk averse investor.

The same is true for Sortino, Kappa, Omega, and Stutzer by virtue of the Equivalence Relation discussed above for Gaussian portfolios.

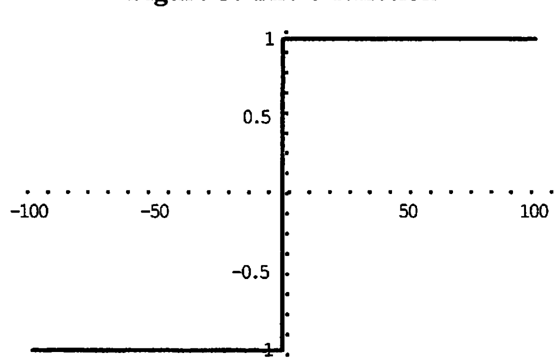
The x-axis shows return in percent, and the y-axis shows the corresponding utility. The q-function associates to all positive returns the same utility +1 and all negative returns the same utility -1. Using the Sharpe ratio (as well as Omega, Sortino, Kappa, or Stutzer) for portfolio selection, in the Case #2 discussed in the paper, is equivalent to implicitly adopting the above utility function. Therefore such investor is not a risk averse inverse. The investor does not value a loss of -100% less than a loss of –1 percent.

The CARA Utility Function

For Gaussian portfolios, the use of the Levy ranking corresponds to the use of the CARA utility function. For non-Gaussian portfolios, this correspondence is broken by the presence of skewness and kurtosis in the portfolio distribution.

Interestingly it is possible to correct the Levy ranking to correct for skewness and kurtosis, and we obtain the

Figure 3: The θ-function



"corrected" Levy ranking formula

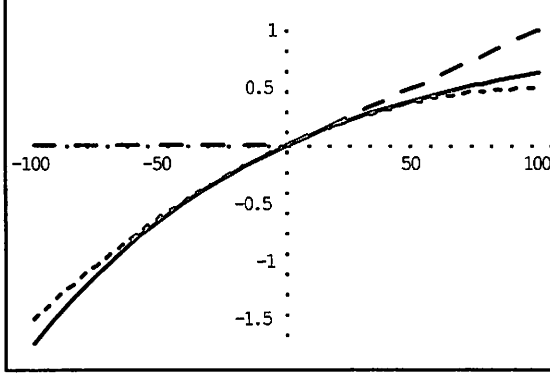
$$[return] + m/2[risk]^2 - m^2/6\,[risk]^3\,[skewness] + m^3/720\,[risk]^4\,[kurtosis] \quad \text{("corrected" Levy)}. \qquad (8)$$

(*[kurtosis]* here is the reduced kurtosis, and it is zero for a Gaussian.) This formula follows from the Correspondence Relation by using CARA and expanding the integrand in Taylor series to the 4th order.

The CARA* Utility Function

As an exercise we considered a new utility function that we called CARA*. It consists of an approximation to CARA at second order. Figure 4 below shows that CARA* approximates very well CARA for returns in range (-100%, +100%).

Figure 4: CARA Utility Function



The continuous line shows the CARA utility function, the x-axis shows return in percent, and the y-axis shows the corresponding utility. The small dashed line shows the CARA* utility function, defined as an approximation to CARA correct at the second order in Taylor. The large dashed line shows the CARA** utility function, which is equivalent to CARA* for negative returns but linear for positive returns. These three utility functions correspond to being risk averse. CARA and CARA* are very similar for practical purposes, while CARA** tends to overvalue positive returns, and thus undervalue risk, when compared with CARA or CARA*. Counter intuitively CARA** corresponds to using downside risk in place of standard deviation for [risk] in the Levy ranking formula - [return] - m/2 [risk].

The use of the Levy ranking without correction, in the case of non-Gaussian distributions, corresponds to the implicit choice of the CARA* utility function.

<u>The CARA** Utility Function</u>

As another exercise we asked which utility function would correspond to the use of the Levy ranking if the measure of *[risk]* were replaced by *[downside risk]*. We found that, counter intuitively, this ranking scheme corresponds to what we call CARA** utility function, *i.e.*, a utility function that is a quadratic approximation for CARA for negative returns and linear for positive returns. In other words CARA** is very close to CARA for negative returns (losses) but weights positive returns (gains) more than CARA does. Therefore the substitution of *[risk]* with *[downside risk]* in the Levy ranking has the effect of giving a premium to large potential gains instead of penalizing large positive losses as naively expected.

CONCLUSIONS

In this paper we have studied the issue of portfolio ranking for portfolio selection. We can summarize our conclusions in the block diagram of Figure 5.

We have distinguished two main cases: when the investment amount is to be determined and one is free to distribute funds between the selected portfolio and the risk-free asset; when there is a predetermined amount that has to be invested in the selected portfolio. In the former case, if the portfolios exhibit a Gaussian distribution, then it is correct to rank them using the Sharpe ratio (Sortino, Omega, Kappa or Stutzer would produce the same ranking). If portfolios do not have a Gaussian distribution, the above ranking schemes correspond to different definitions of risk. In the second case (for a predetermined amount), none of those ranking schemes are appropriate. For Gaussian distributions, a rational risk averse investor using the CARA utility function would rank portfolios using the Levy formula. For non-Gaussian distributions he should use our "corrected" Levy formula:

$$[return] + m/2[risk]^2 - m^2/6\,[risk]^3\,[skewness] + m^3/720\,[risk]^4\,[kurtosis] \quad \text{("Corrected" Levy)} \qquad (9)$$

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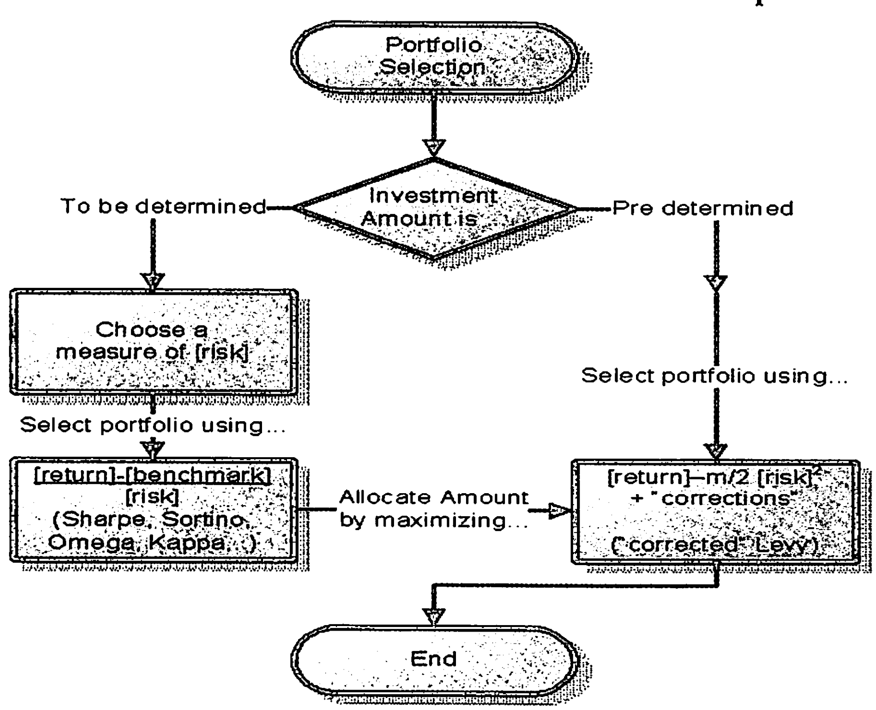
Figure 5: This Flow Chart Represents the Correct Decision Process for Portfolio Selection as Discussed in this Paper



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