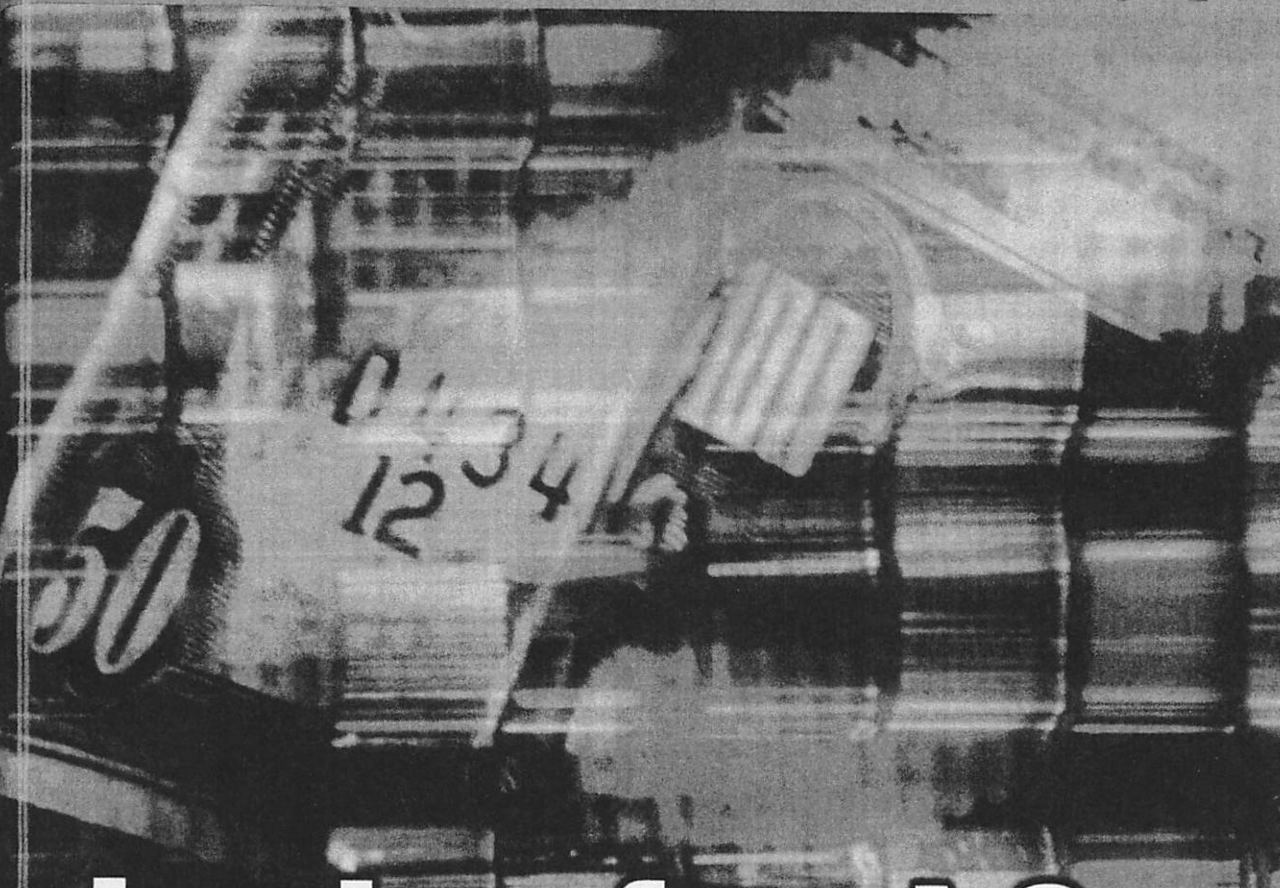
ELSEVIER FINANCE



hedge fund & investment management

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B H

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Chapter 9

On ranking schemes and portfolio selection

MASSIMO DI PIERRO AND JACK W. MOSEVICH

ABSTRACT

There are now in use several risk-return indicators, which are utilized to rank historical returns of portfolios. Some popular ones are the Sharpe Ratio, the Sortino Ratio, Omega and the Stutzer Index, among others. It is well known that portfolio log-returns, especially of alternative assets, are not normally (Gaussian) distributed. This is the reason for the development of indicators other than the Sharpe Ratio. The purpose of this paper is to evaluate the relationships between these indicators for both Gaussian and non-Gaussian distributions. We prove mathematically that rankings are essentially the same for these indicators in a Gaussian environment, and different in a non-Gaussian one, which is as it should be. We are able to compute an implied utility function for the indicators and find that it is the same for all of them, something not very intuitive. We then propose a utility function, which corresponds more with what we expect investors to desire. We conclude by showing how to relate our results to the Markowitz MPT.

9.1 INTRODUCTION

In this paper we discuss different criteria for ranking portfolios including the Sharpe ratio (Sharpe, 1964), the Sortino ratio (Sortino and Van Der Meer, 1991; Sortino and Price, 1994; Sortino and Forsey, 1996), the kappa ratio (Kaplan and Knowles, 2004), the omega ratio (Shadwick and Keating, 2002; Sortino, 2001; Wilmott, 2000) and the Stutzer index (Amenc, Malaise, Martellini and Vaisse). We prove that in a world where portfolio returns are Gaussian distributions, all of the above ranking systems are equivalent in the sense that although they produce different numbers they will produce the same ranking order. We also prove that all of the above ranking systems implicitly assume a non-natural utility function that attributes the same utility to any positive return (utility equals to $+1$) and to all negative returns (utility equals to -1).

We propose a more natural utility function from which we derive a different ranking system for Gaussian portfolios that is not equivalent to the Sharpe ratio or any of the other rankings considered. Using the Berry–Esseen theorem we prove that our ranking system, embodied in equation (24), is applicable to portfolios with non-Gaussian returns under the condition that one plans to hold the portfolio for a sufficiently long time.

Finally we show how to apply our findings to Markowitz' Modern Portfolio Theory (Markowitz, 1952; Wilmott, 2000).

9.2 CONVENTIONS AND DEFINITIONS

Given a portfolio A whose historic values are $\{S_i\}$ we will indicate with $p(x) : R \to R^*$ the probability mass function of each of the random variables $x_i = \log(S_{i+1}/S_i)$. The probability mass function is normalized to 1. We also define as $F(x) = \int_{-\infty}^{x} p(z)\mathrm{d}z$ the usual cumulative distribution.

Throughout this paper we will also assume that r is the risk-free interest rate.

Definition: (*Ranking*). We define a raking as a functional $R(p) : R \times R^* \to R$ that maps the probability mass function p associated to a portfolio into a real number.

Definition: (*Equivalence*). We say that two rankings R_1 and R_2 are equivalent in a domain D if there exists a monotonic increasing function h such that for every probability mass function p in the domain D, $R_1(p) = h(R_2(p))$.

If two rankings are equivalent we will use the notation

$$R_1 \sim R_2$$

The equivalence relation as defined is symmetric and transitive.

Definition: (*Sharpe ratio*). Given a portfolio characterized by a probability mass function $p(x)$ the Sharpe (1964) ratio is defined as

$$R_{\text{Sharpe}}(p) \stackrel{\text{def}}{=} \frac{\mu - r}{\sigma} \tag{1}$$

where

$$\mu = \int_{-\infty}^{+\infty} xp(x)\mathrm{d}x \tag{2}$$

$$\sigma = \sqrt{\int_{-\infty}^{+\infty} (x - \mu)^2\, p(x)\, \mathrm{d}x} \tag{3}$$

Herein we denote the Sharpe ratio by y, with no explicit reference to the risk-free rate r. Note that in the equations which follow, the letter r can also represent a minimal acceptable return.

Definition: (*Sortino ratio*). Given a portfolio characterized by a probability mass function $p(x)$ the Sortino ratio (Sortino and Van Der Meer, 1991; Sortino and Price, 1994; Sortino and Forsey, 1996) is defined as

$$R_{\text{Sortino}}(p) \stackrel{\text{def}}{=} \frac{\mu - r}{\sqrt{\int_{-\infty}^{r} (r - x)^2 \, p(x) \, \mathrm{d}x}} \tag{4}$$

Definition: (*Kappa-n ratio*). Given a portfolio characterized by a probability mass function $p(x)$ the kappa ratio (Kaplan and Knowles, 2004) is defined as

$$R_{\text{kappa}-n}(p) \stackrel{\text{def}}{=} \frac{\mu - r}{\left[\int_{-\infty}^{r} (r - x)^n p(x) \mathrm{d}x\right]^{1/n}} \tag{5}$$

Note that for $n = 2$, $R_{\text{kappa}-2}(p, r) \equiv R_{\text{Sortino}}(p, r)$.

Definition: (*Omega ratio*). Given a portfolio characterized by a probability mass function $p(x)$ the omega ratio (Shadwick and Keating, 2002; Sortino, 2001; Wilmott) is defined as

$$R_{\text{Omega}}(p) \stackrel{\text{def}}{=} \frac{\int_{\infty}^{r} (1 - F(x)) \mathrm{d}x}{\int_{-\infty}^{r} F(x) \mathrm{d}x} \tag{6}$$

Definition: (*Stutzer index*). Given a portfolio characterized by a probability mass function $p(x)$ the Stutzer index (Amenc, Malaise, Martellini and Vaisse) is defined as

$$R_{\text{Stutzer}}(p) \stackrel{\text{def}}{=} \lim_{T \to \infty} \frac{-\log F(rT)}{T} \tag{7}$$

The Stutzer index ranks portfolios according to the speed with which the probability of a negative return (when compared with rT) tends to zero when time grows ($T \to \infty$).

9.3 EQUIVALENCE IN A GAUSSIAN WORLD

In this section we will restrict to only domain

$$D = \left\{ p \,|\, p(x) = p_{\text{Gaussian}}(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-(x-\mu)^2/2\sigma^2} \right\} \tag{8}$$

portfolios having a Gaussian distributions of returns.

Theorem 1. $R_{\text{Sortino}} \sim R_{\text{Sharpe}}$.

Proof. By explicit integration[1]

$$R_{\text{Sortino}}(p) = h_1(R_{\text{Sharpe}}(p))$$

Where

$$h_1(y) = \frac{\sqrt{2}y}{\sqrt{1 - \sqrt{2/\pi}\mathrm{e}^{-y^2/2}y + y^2 - (1 + y^2)\operatorname{erf}(y/\sqrt{2})}} \tag{9}$$

Recall that $y = (\mu - r)/\sigma$ is the Sharpe ratio. Since $h_1'(y) > 0$ for every finite real is y proves the equivalence.

Figure 1 shows a plot of h_1 and h_1'. It also shows that compared with the Sharpe ratio, the Sortino is relatively sensitive to changes of y for large values of y, but it becomes insensitive to y for negative values of y.

Theorem 2. $R_{\text{kappa}-n} \sim R_{\text{Sharpe}}$ for every n.

Proof. By explicit integration

$$R_{\text{kappa}-n}(p) = h_2(R_{\text{Sharpe}}(p)) \tag{10}$$

$$h_2(y) = \frac{\pi^{1/(2n)}y}{\left[2^{(n-2)/2}\mathrm{e}^{-y^2/2}g(y)\right]^{1/n}}$$

$$g(y) = \Gamma\left(\frac{1+n}{2}\right)_1F_1\left(\frac{1+n}{2}, \frac{1}{2}, y^2/2\right) - \sqrt{2}y_1F_1\left(\frac{1+n}{2}, \frac{3}{2}, y^2/2\right)$$

and ${}_1F_1(a, b, x)$ is a member of the family of hypergeometric functions. $h_2'(y) > 0$ for every even integer n and every finite real y. Figure 2 shows a plot of h_2 and h_2' for $n = 1, 2, 3$. It also shows how the K_n ratio exhibits the same sensitivity to y as the Sortino does, but the higher the value of n, the lower the sensitivity to y.

Theorem 3. $R_{\text{Omega}} \sim R_{\text{Sharpe}}$.

Proof. By explicit integration

$$R_{\text{Omega}}(p) = h_3(R_{\text{Sharpe}}(p)) \tag{11}$$

$$h_3(y) = 1 + \frac{2y}{\sqrt{2/\pi}e^{-y^2/2} - y + y\operatorname{erf}(y/\sqrt{2})}$$

$h_3'(y) > 0$ for finite real y. Figure 3 shows a plot of h_3 and h_3'.

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[1] $\operatorname{erf}(z) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-x^2}\,\mathrm{d}x$

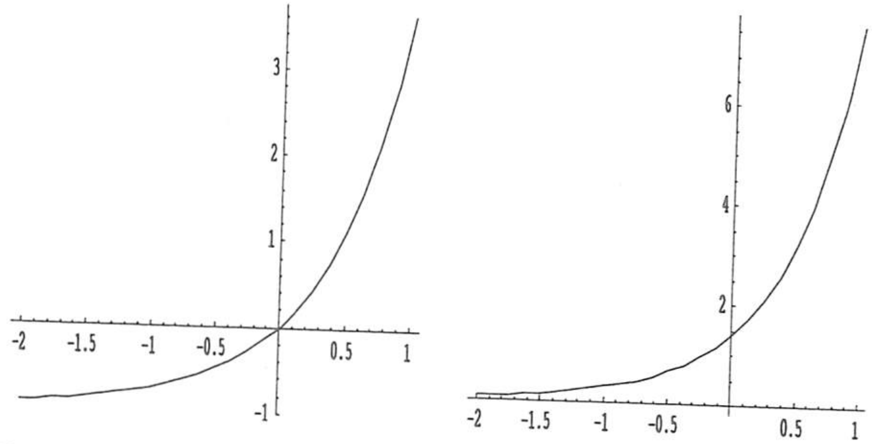

Figure 1 Plot of $h_1(y)$ (left) and $h_1'(y)$ (right). h_1 is the Sortino ratio as function of the Sharpe ratio for a portfolio with Gaussian returns

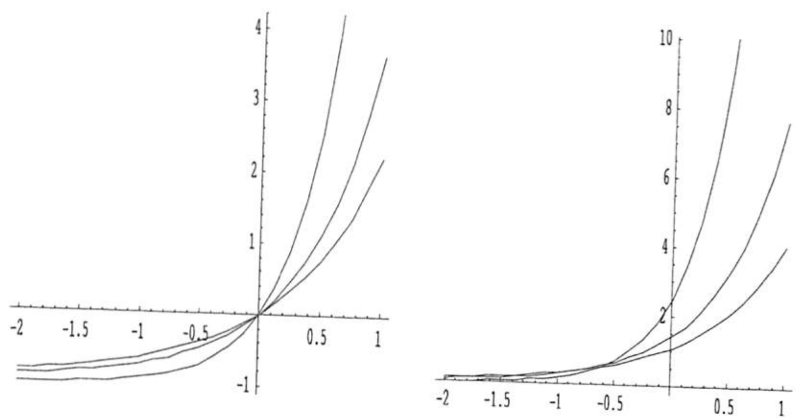

Figure 2 Plot of $h_2(y)$ (left) and $h_2'(y)$ (right) for $n = 1, 2, 3$

Theorem 4. $R_{\text{Stutzer}} \sim R_{\text{Sharpe}}$ if and only if $y > 0$.

Proof. By explicit integration

$$R_{\text{Stutzer}}(p) = h_4(R_{\text{Sharpe}}(p))$$

$$h_4(y) = \begin{Bmatrix} y^2/2 & \text{for } y > 0 \\ 0 & \text{for } y \leq 0 \end{Bmatrix} \tag{12}$$

$h_4'(y) > 0$ for real $y > 0$. Figure 4 shows a plot of h_4 and h'_4. For a portfolio with Gaussian mass function, the Stutzer index is unable to rank portfolios with negative y.

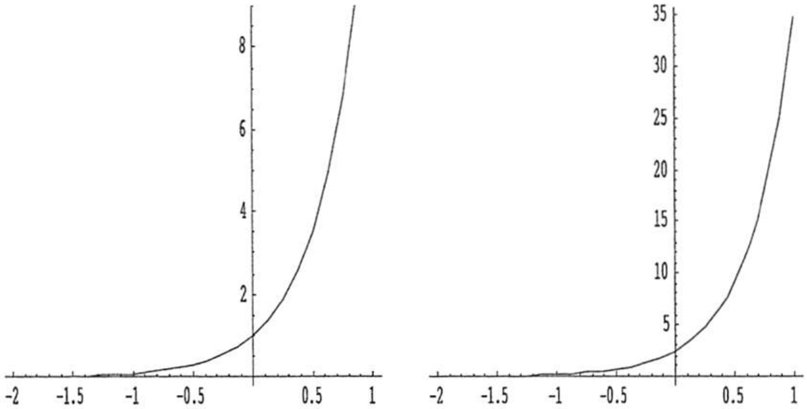

Figure 3 Plot of $h_3(y)$ (left) and h'_3 (right). h_3 is the omega ratio as function of the Sharpe ratio for a portfolio with Gaussian returns

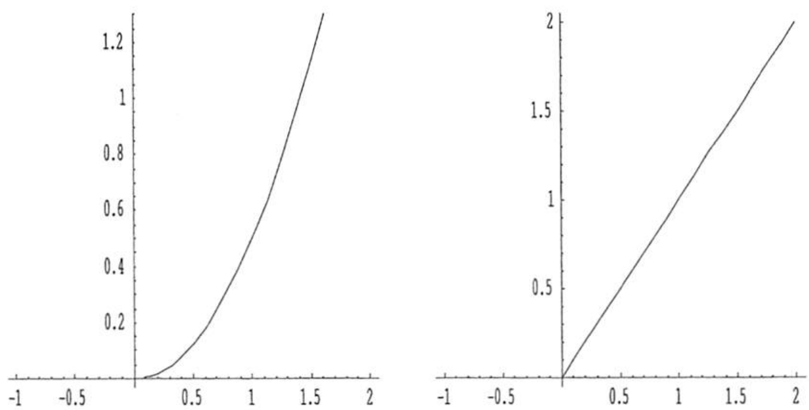

Figure 4 Plot of $h_4(y)$ (left) and h'_4 (right). h'_4 is the Stutzer index as function of the Sharpe ratio for a portfolio with Gaussian returns

So far, we have shown how in the domain of portfolios with Gaussian returns

$$R_{\text{Sharpe}} \sim R_{\text{Sortino}} \sim R_{\text{Kappa}-\text{n}} \sim R_{\text{Omega}} \tag{13}$$

and in the subdomain with portfolios with positive R_{Sharpe}

$$R_{\text{Sharpe}} \sim R_{\text{Stutzer}} \tag{14}$$

Moreover, we have shown how these indices tend to be more and more sensitive to y and for larger positive y and less and less sensitive for more negative y.

Three questions will be addressed in the following sections:

- Given that all of the above systems are equivalent, are they good measures to rank portfolios?
- If not, what is a better system to rank portfolios?
- How do we extend these results to portfolios with non-Gaussian returns?

9.4 RANKING AND RISK AVERSION

Consider a situation where the risk-free interest rate is $r = 5\%$ and four possible future scenarios for a portfolio:

- #1: The portfolio out-performs r with a return $x = 20\%$
- #2: The portfolio out-performs r with a return of $x = 10\%$
- #3: The portfolio under-performs r with a return of $x = 3\%$
- #4: The portfolio under-performs r with a return of $x = -2\%$.

It is clear that we prefer #1 to #2, #2 to #3 and #3 to #4 but how do we quantify this preference? How much more do we prefer #1 to #3 when compared to #2? How bad is #4 when compared with #3? The answers to these questions have nothing to do with probability (we are not discussing here the likelihood of one scenario over the other) but have to do with subjective choice and one's perception of risk.

This choice is equivalent to the choice of an implied utility function that we will indicate by $W(x)$. It returns one's subjective utility of the scenario in which the portfolio has a fixed return x. For logical reasons, we value higher returns more than lower returns. So $W'(x) \geq 0$ and this is the only *a priori* condition we wish to impose.

Given a utility function it is natural to rank a portfolio by evaluating a weighted average of $W(x)$ over all possible future scenarios. The weight factor is the probability of a future scenario with return x. This induced ranking can be expressed as

$$R_W(p) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} W(x)p(x)\,\mathrm{d}x \tag{15}$$

Now the question becomes: in a Gaussian world, which choice of $W(x)$ produces a ranking equivalent to the Sharpe ratio (and all the other rankings equivalent to the Sharpe ratio)?

Surprisingly, the answer is

$$W(x) = W_{\text{naive}}(x) \stackrel{\text{def}}{=} (\mathrm{x} - r)/|x - r| \tag{16}$$

which implies

$$R_{\text{naive}}(x) = \int_{-\infty}^{+\infty} W_{\text{naive}}(x)\,p(x)\,\mathrm{d}x$$

$$= \int_{-\infty}^{+\infty} \frac{(x-r)\mathrm{e}^{-(x-\mu)^2/2\sigma^2}}{|x-r|\;\sqrt{2\pi}\sigma}\,\mathrm{d}x$$

$$= \operatorname{erf}((\mu - r)/\sigma\sqrt{2})) \qquad (17)$$

and

$$R_{\text{naive}}(p) = h_5(R_{\text{Sharpe}}(p)) \qquad (18)$$

$$h_5(y) = \operatorname{erf}(y/\sqrt{2})$$

We just proved that for portfolios with Gaussian returns $h_5'(y) > 0$.

Theorem 1. $R_{\text{naive}} \sim R_{\text{Sharpe}}$.

This finding is surprising because it implies that in a Gaussian world, ranking portfolios according to the Sharpe ratio, the Sortino, the Kappa, the Omega or the Stutzer index is equivalent to having the utility function in equation (16). This utility function is plotted in Figure 5.

W_{naive} is not a risk averse utility function. Referring to the examples of the four scenarios at the beginning of the section this naive utility function implies that we like scenarios #1 and #2 equally (utility $W = +1$) and we dislike scenarios #3 and #4 equally (utility $W = -1$).

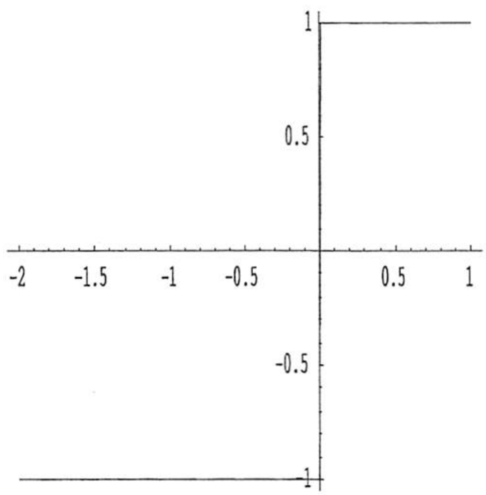

Figure 5 Plot of $W_{\text{naive}}(x)$, the utility function (function of the portfolio return) that is equivalent to ranking portfolios using the Sharpe ratio

9.5 A BETTER UTILITY FUNCTION

Clearly, the utility function induced by the Sharpe ratio is not natural and does not capture the natural risk aversion of many investors. There is an extensive literature in economics describing more rational choices for utility functions. Because this is a subjective choice we cannot claim that any one utility function is better than all others, but we can select a utility function that is more natural than the one in equation (16) and exhibits the following desirable characteristics:

- $W'(x) > 0$; the higher the return of a given scenario the higher the utility of the scenario.
- $W(r) = 0$; a scenario in which the return is the same as the risk-free rate has zero utility.
- $\lim_{x\to\infty} W'(x) = 0$; we became insensitive to x for large positive returns.
- $\lim_{x\to-\infty} W'(x) = \infty$; we are extremely sensitive to x for large negative returns.

A utility function that exhibits all of the above characteristics is the constant absolute risk aversion (CARA) (Wilmott)

$$W_{\mathrm{CARA}}(x) \stackrel{\mathrm{def}}{=} -\mathrm{e}^{-m(x-r)} \tag{19}$$

Here k is a subjective number that measures one's risk aversion. The larger the k, the more risk averse one is. The CARA utility function $W_{\mathrm{CARA}}(x)$, is shown in Figure 6.

When we substitute equation (19) into equation (15) for a Gaussian $p(x)$ we obtain

$$R_{\mathrm{CARA}}(p) = \int_{-\infty}^{+\infty} W_{\mathrm{CARA}}(x)p(x)\,\mathrm{d}x \tag{20}$$

$$= \int_{-\infty}^{+\infty} -\mathrm{e}^{m(r-x)} \frac{\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\,\mathrm{d}x \tag{21}$$

$$= -\mathrm{e}^{m(r-\mu)+m^2\sigma^2/2} \tag{22}$$

In order to have the ranking to be a pure number in the range $(-\infty, +\infty)$, we define

$$R_{\mathrm{best}}(p) \stackrel{\mathrm{def}}{=} \mu/r - 1 - m\sigma^2/(2r) \tag{23}$$

which is equivalent to $R_{\mathrm{CARA}}(p)$ because

$$R_{\mathrm{best}}(p) = h_6(R_{\mathrm{CARA}}(p)) \tag{24}$$

$$h_6(z) = -\log(-z)/(mr) \tag{25}$$

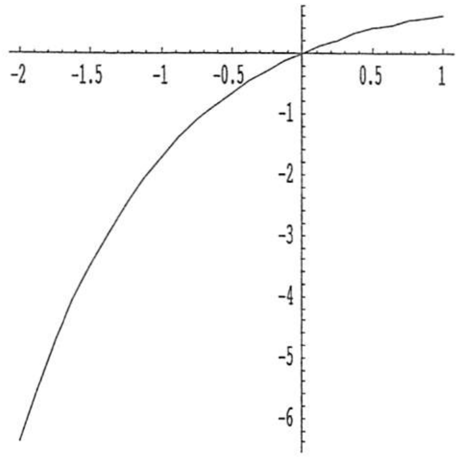

Figure 6 Plot of $W_{CARA}(x)$, the CARA utility function used to derive $R_{best}(p) \sim R_{CARA}(p)$

and $h'_6(z) > 0$ for $mr > 0$ and $z < 0$.

Note that, for the risk-free asset, $R_{best} \simeq 0$. "Good" portfolios rank above 0 and "bad" portfolios rank below 0.

It is important to notice how the Sharp ratio, as well as the Sortino, the kappa, the omega and the Stutzer index do not depend on any subjective parameters except for minimal acceptable return r (and n in the case of kappa), while Equation (23) depends also on the subjective value of m. This is not surprising since the latter incorporates a scale, represented by m, that encodes information about how better scenario #1 is when compared with scenario #2. The Sharpe ranking does not incorporate any information about this scale and it assumes that scenario #1 is as good as scenario #2, and scenario #3 is as bad as scenario #4. The introduction of at least one parameter (in our case m) is necessary to quantify the cost of taking a risk. The higher the value of m the higher will be the cost of risk.

The ranking scheme R_{best} is not equivalent to R_{Sharpe} as shown for the following portfolios (Gaussian returns, $r = 5\%$ and $m = 2, 4, 8$, the latter choices will be explained later).

Note that in all the schemes portfolio, E ranks better than A (because it has the same risk but higher return) and I (because it has the same return but lower risk). For the same reasons F ranks better than B and J, G than C and K, and H than D and L as was expected. Nevertheless, the relative ranking of portfolios is different for different schemes and different choices of m:

- $R_{Sharpe} \Rightarrow$ E, A, F, I, G, B, J, H, C, K, D, L
- R_{best} and $m = 2 \Rightarrow$ H, L, D, G, K, C, F, J, B, E, I, A
- R_{best} and $m = 4 \Rightarrow$ H, G, D, L, K, C, F, J, B, E, I, A
- R_{best} and $m = 8 \Rightarrow$ F, E, B, I, J, A, G, C, K, H, D, L

	μ	σ	R_{Sharpe}	R_{best}	R_{best}	R_{best}
				$(m = 2)$	$(m = 4)$	$(m = 8)$
A	16%	9%	1.22	2.04	1.88	1.55
B	21%	14%	1.14	2.81	2.42	1.63
C	28%	21%	1.10	3.72	2.82	1.07
D	32%	25%	1.08	4.15	2.90	0.40
E	17%	9%	1.33	2.24	2.08	1.75
F	22%	14%	1.21	3.01	2.62	1.83
G	29%	21%	1.14	3.92	3.04	1.27
H	33%	25%	1.12	4.35	3.10	0.60
I	17%	10%	1.20	2.20	2.00	1.60
J	22%	15%	1.13	2.95	2.50	1.60
K	29%	22%	1.09	3.83	2.86	0.93
L	33%	26%	1.08	4.25	2.90	0.19

(26)

According to CARA for $m = 4$ the best portfolio is H and according to Sharpe it is E.

9.6 EXTENSION TO NON-GAUSSIAN DISTRIBUTIONS

In the real world the random variables x_i are not Gaussian and one may wonder how this affects our conclusions.

First of all the statement that the Sharpe ratio, the Sortino, the kappa, the omega and the Stutzer index are equivalent is not true any more and each corresponds to a different implicit choice of a utility function. One cannot answer the question to which one is better because there is no objective benchmark any more.

Anyway the real world is "close" to Gaussian if returns are computed over relatively long time periods (this is shown later) and all of these ranking systems are inappropriate in the Gaussian limit. Therefore, there is no reason to believe they should be appropriate for non-Gaussian or close-to-Gaussian distributions.

On the other hand, despite the fact that in the preceding section, equations (20)–(22) and equation (23) are derived assuming a Gaussian $p(x)$, the *ranking induced by equation (23) is still correct for non-Gaussian distributions, providing that one plans to hold portfolio a long enough time.*

First of all it is important to realize that in the Gaussian world equations (20–22) do not depend on the time scale, since the probability distribution for the 1-, the 2- or the 100-day return is always Gaussian. In a non-Gaussian world the probability distribution for 1-day returns is different than the probability distribution for 2-day returns, etc. In order to take this into account we define $p(x)$ as the probability mass function for 1-day returns and, in general, $p_T(X)$ as the probability mass function of

T-day returns where $X = \log(S_T/S_0) = \sum_{i=0}^{i<T} x_i$ and $x_i = \log(S_{i+1}/S_i)$. We will assume all the random variables x_i are independent and follow the distribution p. We also define μ and σ as the mean and standard deviation of p.

We can now rewrite equation (15) as the weighted utility at the end of period T

$$R_W(p, T) = \int_{-\infty}^{+\infty} W(X/T)\, p_T(X)\, \mathrm{d}X \tag{27}$$

Owing to the Central Limit Theorem and the Berry–Esseen Theorem, when $T \to \infty$, p_T approaches a Gaussian distribution with mean μT and standard deviation $\sigma\sqrt{T}$. This is demonstrated in Figure 7 for an initial triangular distribution. Therefore for our choice $W = W_{\mathrm{CARA}}$, equation (27) implies

$$\lim_{T\to\infty} R_{\mathrm{CARA}}(p, T) = -\mathrm{e}^{m(r-\mu)+m^2\sigma^2/2} \sim R_{\mathrm{best}}(p) \tag{28}$$

It remains to address how fast R_{CARA} approaches the limit. From the Berry–Esseen theorem it follows that:

Lemma 1. For every probability mass function p and every $\varepsilon > 0$ there exists a $c > 0$ such that

$$\left| R_{\mathrm{CARA}}(p, T) + \mathrm{e}^{m(r-\mu)+m^2\sigma^2/2} \right| < c/\sqrt{T} + \varepsilon \tag{29}$$

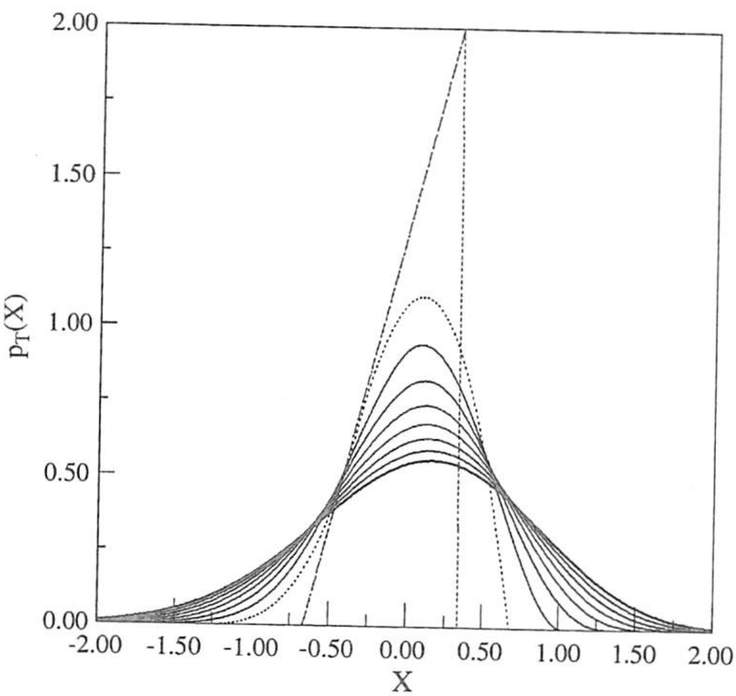

Figure 7 $p_T(x)$ for $T = 1, \ldots, 10$ is shown given a triangular distribution for $p(x)$ (the dashed line). When $T \to \infty$, $p_T(x)$ approaches a Gaussian distribution

where c depends on ε and the shape of p.

Finally from equation (28) and the above lemma we conclude that:

Proposition 2. In the general case of portfolios with random returns having non-Gaussian distributions one should rank the portfolios by explicit integration of equation (27). Nevertheless, for our choice of the CARA utility function and for a long holding period T, this is equivalent to ranking the portfolios using R_{best}, equation (23). The error incurred is proportional to $1/\sqrt{T}$.

9.7 MARKET DETERMINATION OF *m*

In our notation m is a positive number that encodes the investor's risk tolerance. A low value of m (close to zero) indicates a high tolerance of risk, while a high value of m indicates a low tolerance of risk (risk aversion).

Despite the fact that m is subjective one can ask if there is something like an "average market value" for m. In order to address this question we considered the Dow Jones industrial (DJI) average index in the range from Oct 1984 until Oct 2004 and we compute the average yearly return μ and the average yearly volatility σ (standard deviation) of the weekly lognormal returns. We find

$$\mu = 0.1076 (= 10.76\%) \tag{30}$$

$$\sigma = 0.1620 (= 16.20\%) \tag{31}$$

we then assume that the DJI has the same ranking as a risk-free interest (= 0)

$$R_{best}(DJI) = \frac{\mu}{r} - 1 - m\frac{\sigma^2}{2r} = 0 \tag{32}$$

From equation (32) and a reasonable guess $r \simeq 0.05$ (5%) we obtain

$$m \simeq 4 \tag{33}$$

Therefore, in this paper we consider empirical values of m in the range from 2 (for a risk lover investor) to 8 (for a very risk averse investor) and a typical value $m = 4$ for an average investor.

9.8 MODERN PORTFOLIO THEORY

Finally we wish to clarify the role that equation (23) plays in the context of Markowitz' modern portfolio theory (MPT) (Markowitz, 1952; Wilmott, 2000).

In a typical problem one is given a set of N assets, μ_i being the expected return from asset i and σ_{ij}, the covariance between asset i and j. The risk-free rate is r. What is the optimal portfolio?

The MPT establishes that the optimal portfolio is a combination of the risk-free asset (with weight α) and the Markowitz portfolio (with weight $1 - \alpha$). The

Markowitz portfolio is a linear combination of the assets (excluding the risk-free one) with weights equal to

$$w_i \stackrel{\text{def}}{=} \sum_j (\sigma^{-1})_{ij}(\mu_j - r) \tag{34}$$

the mean and variance of the Markowitz portfolio are given by

$$\mu_M \stackrel{\text{def}}{=} \sum_j w_i \mu_i \text{ and } \sigma_M \stackrel{\text{def}}{=} \sqrt{\sum_j w_i w_j \sigma_{ij}} \tag{35}$$

The MPT decouples the problem of finding the optimal combination of risky assets with that of finding the optimal combination of risk-free asset and risky assets. The MPT solves the first problem but the solution of the second problem is not uniquely determined because it leaves α undetermined.

Since α is not determined by the Markowitz's method, its value is subjective and, in its own way, α measures the attitude towards risk of the investor. We see α is related to our value of m.

Let M_α be a portfolio, which is a linear combination of the risk-free asset with weight α and the Markowitz portfolio (as computed by the MPT) with weight $(1 - \alpha)$. The probability mass function associated to this portfolio is

$$p_\alpha(x) = \alpha\delta(x - r) + (1 - \alpha)\frac{1}{\sqrt{2\pi}\sigma_M} e^{-(x-\mu_M)^2/2\sigma_M^2} \tag{36}$$

where δ is the Dirac delta function. We can now determine α by maximizing $R_{\text{best}}(p_\alpha)$. This procedure is represented graphically in Figure 8. The solution can be found by explicit computation as stated in the following:

Theorem 1. Given a risk-free rate r and a Markowitz portfolio (μ_M, σ_M), according to the R_{best} ranking, the optimal portfolio consists of holding a fraction α of the risk-free asset where

$$\alpha = 1 - \frac{\mu_M - r}{m\sigma_M^2} \tag{37}$$

and a fraction $(1 - \alpha)$ of the Markowitz portfolio. This optimal portfolio has an expected average return and volatility given by

$$\mu_{\text{best}} = r + \frac{(\mu_M - r)^2}{m\sigma_M^2} \text{ and } \sigma_{\text{best}} = \frac{\mu_M - r}{m\sigma_M} \tag{38}$$

Note that the Sharpe ratio cannot be used to solve the problem of determining α for two reasons: (1) the Sharpe ratio is undetermined for the risk-free asset; and (2) in the return versus risk plot the indifference curves associated with constant rankings are straight lines, therefore they cannot be tangent to the Markowitz line parametrized by α.

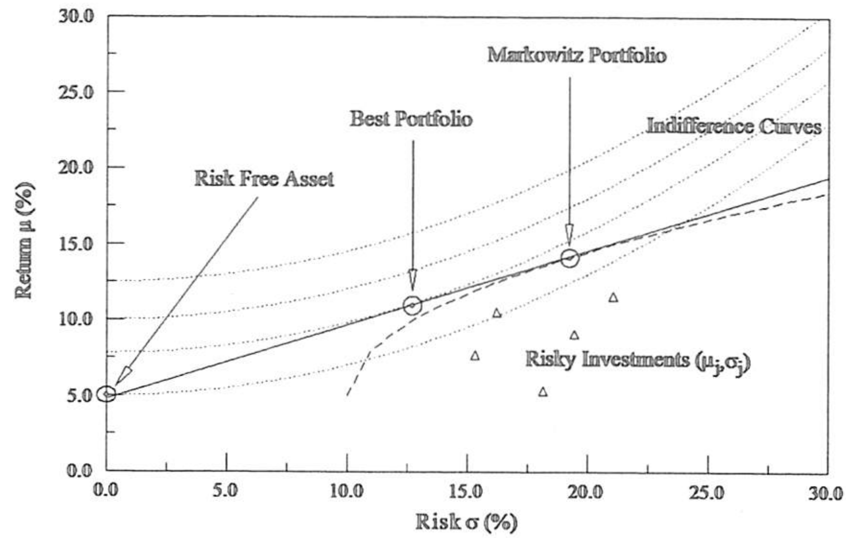

Figure 8 The MPT on a μ, σ plane is shown, where the ranking $R_{best}(p)$ is used to determine the investor's indifference curves. All portfolios on the line are equivalent according to MPT. The concave parabolas represent indifference curves (sets of portfolios having the same ranking). The higher the parabola, the higher the ranking. The best portfolio can be determined by finding the indifference curve tangent to the line or indifference curves. All portfolios on the line are equivalent according to MPT. The concave parabolas represent indifference curves (sets of portfolios having the same ranking). The higher the parabola, the higher the ranking. The best portfolio can be determined by finding the indifference curve tangent to the line figure

9.9 CONCLUSIONS

In this paper we proved that using the Sharpe ratio, the Sortino, the kappa, the omega or the Stutzer index to rank portfolios are equivalent choices when portfolios have Gaussian returns (that eventually is true for sufficiently long time T). Nevertheless, all of these ranking systems implicitly assume a utility function that is not consistent with the risk aversion of investors. Moreover these ranking schemes are not useful for many important practical applications (such as finding indifference curves for use in MPT).

With our choice of the CARA utility function $W(x) = -e^{-m(x-r)}$ (where r is the risk-free rate and m parametrizes our risk attitude) we determined a ranking formula

$$R_{best}(p) = \mu/r - 1 - m\sigma^2/(2r) \tag{39}$$

that is applicable to portfolios with Gaussian and non-Gaussian returns. In the latter case the formula is valid but introduces a numerical error in the ranking that is proportional to $T^{-1/2}$. T is the time one is planning to hold the portfolio. This error can be arbitrarily reduced by holding the portfolio for a sufficiently long time.

Finally, we performed an analysis of the *DJI* average index (data from 1984 until today) and determined an average market value for $m \simeq 4$. Further studies on the time dependence of m and its correlation with other market indicators are required.

According to the EDHEC (Amenc, Malaise, Martellini and Vaisse) 69% of European Hedge Funds reports the Sharpe ratio to their investors and 22% report the Sortino ratio. We believe these numbers are misleading and should be used with caution. In particular:

- One should not at all rank portfolios that are correlated, in this case one should use the MPT or the CAPM models.
- If one is considering a set of uncorrelated portfolios (or portfolios with unknown correlations) in order to select the best and invest funds in both the selected portfolio and the risk-free asset, one should use the Sharpe ratio to rank the portfolios. In this case one also needs to select a utility function to decide how to partition the funds between the selected portfolio and the risk-free asset.
- If one is considering a set of uncorrelated portfolios (or portfolios with unknown correlations) in order to invest a fixed amount of money in the selected portfolio, one should use equation (39).

Some of the ideas discussed in this paper are implemented in the form of computer programs and can be accessed through the web page:
http://www.metacryption.com/schemes.html

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